The Gap of the Graph of a Matrix

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ABSTRACT

Assume that A is a liner transformation from C^n into C^n. The gap of A is defined as the gap between two subspaces G_A and G_0, the graph of A and the graph of zero. We prove that the gap of A is $\theta(A) = \|A\|/\sqrt{1+\|A\|^2}$. Here $\|A\|$ is the uniform norm of A.

INTRODUCTION

Let V be an n-dimensional normed linear space over C on which the norm is defined by means of an inner product. Let $A : V \to V$ be a linear transformation, A^* be its adjoint, and $\sigma(A)$ be its spectrum. P is a positive operator, written $P \geqslant 0$, in case $P = P^*$ and $\langle Px, x \rangle \geqslant 0$ for all $x \in V$.

The graph of A is defined to be

$$G_A = \left\{ \begin{pmatrix} x \\ Ax \end{pmatrix} : x \in V \right\}.$$

Here the norm of A is the uniform norm,

$$\|A\| = \sup_{\lambda \in \sigma(A^*A)} \sqrt{\lambda}.$$

We assume that A is represented by an $n \times n$ matrix with respect to the standard normal basis. For a subspace N in C^n ($n \geqslant 2$) we will let Π_N

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denote the orthogonal projection onto N. For two subspaces M and N in C^n we let $\theta(M, N) = \|\Pi_M - \Pi_N\|$, which is a metric on the set of all subspaces of C^n with $\theta(M, N) \leqslant 1$ for all M and N. See Gohberg, Lancaster, and Rodman [1].

The gap of the matrix A, $\theta(A)$, is defined to be the gap between the two subspaces G_A and G_0 in $C^n \oplus C^n$.

THE THEOREM AND RESULTS

We want to compute the gap of the $n \times n$ matrix A. Let

$$\Pi_A = \begin{pmatrix} (I + A^*A)^{-1} & (I + A^*A)^{-1} A^* \\ A(I + A^*A)^{-1} & A(I + A^*A)^{-1} A^* \end{pmatrix}.$$

Then

$$\Pi_A \begin{pmatrix} x \\ Ax \end{pmatrix} = \begin{pmatrix} x \\ Ax \end{pmatrix} \quad \text{and} \quad \Pi_A \begin{pmatrix} -A^* x \\ x \end{pmatrix} = 0.$$

Thus, Π_A is the orthogonal projection onto G_A. The matrix A may be written in the form $A = UP$, where U is a unitary matrix and $P \geqslant 0$. See Horn and Johnson [2]. The matrix P is always uniquely determined as $P = (AA^*)^{1/2}$.

First consider the case of positive matrices. If $P \geqslant 0$, then $P = P^*$ and

$$\Pi_P = \begin{pmatrix} (I + P^2)^{-1} & P(I + P^2)^{-1} \\ P(I + P^2)^{-1} & P^2(I + P^2)^{-1} \end{pmatrix}.$$

Let $Q + (\Pi_P - \Pi_0)^*(\Pi_P - \Pi_0)$. Then $Q = P^2(I + P^2)^{-1}$, and

$$\sigma(Q) = \left\{ \frac{\lambda^2}{1 + \lambda^2} : \lambda \in \sigma(P) \right\}.$$

Therefore

$$\theta(P) = \|\Pi_P - \Pi_0\| = \sup_{\lambda \in \sigma(P)} \frac{|\lambda|}{\sqrt{1 + |\lambda|^2}}.$$

Because $x/\sqrt{1+x^2}$ is an increasing function for $x \geqslant 0$, and $\|P\| = \sup_{\lambda \in \sigma(P)} |\lambda|$, we have

$$\theta(P) = \frac{\|P\|}{\sqrt{1+\|P\|^2}}.$$

Now consider an arbitrary matrix A; then because

$$\Pi_A - \Pi_0 = \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix} (\Pi_P - \Pi_0) \begin{pmatrix} I & 0 \\ 0 & U^* \end{pmatrix},$$

we know that $\Pi_A - \Pi_0$ and $\Pi_P - \Pi_0$ are unitarily equivalent. Therefore

$$\|\Pi_A - \Pi_0\| = \|\Pi_P - \Pi_0\|.$$

Noting that $\|A\| = \|P\|$, we have our main result.

THEOREM. *Let A be an $n \times n$ matrix. Then the gap of A is*

$$\theta(A) = \frac{\|A\|}{\sqrt{1+\|A\|^2}}.$$

Moreover $\theta(UA) = \theta(AU) = \theta(A)$ for every unitary matrix U.

The following results are immediate:

(1) $\theta(A) < 1$ for every matrix A.
(2) If A is a unitary matrix, then $\theta(A) = \sqrt{2}/2$.
(3) $\theta(A) = \theta(A^*)$.

REFERENCES

1 I. Gohberg, P. Lancaster, and L. Rodman, *Invariant Subspaces of Matrices with Applications*, Wiley, New York, 1986.
2 R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge U.P., New York, 1985.

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